Bayesian Design of Reliability Experiments

(Meeker and Zhang, EQR 2008)

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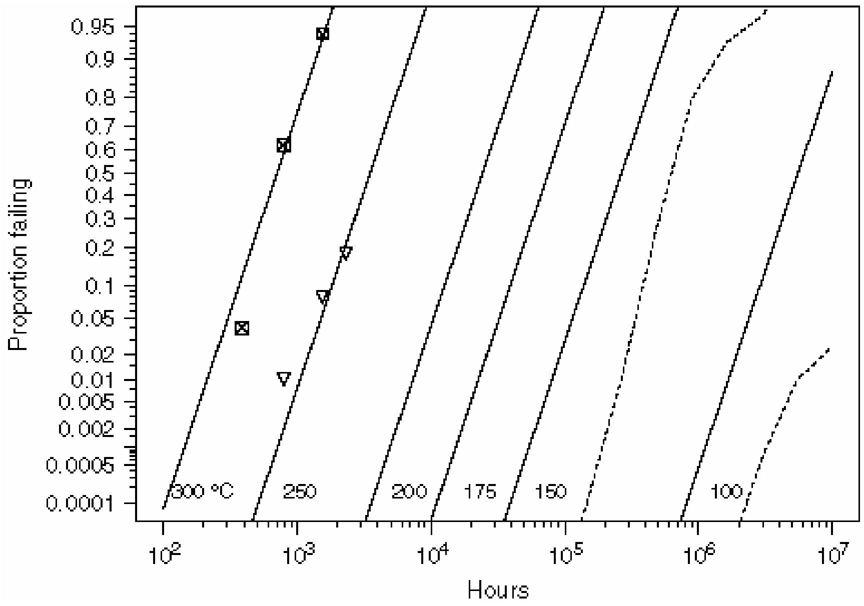
Outline

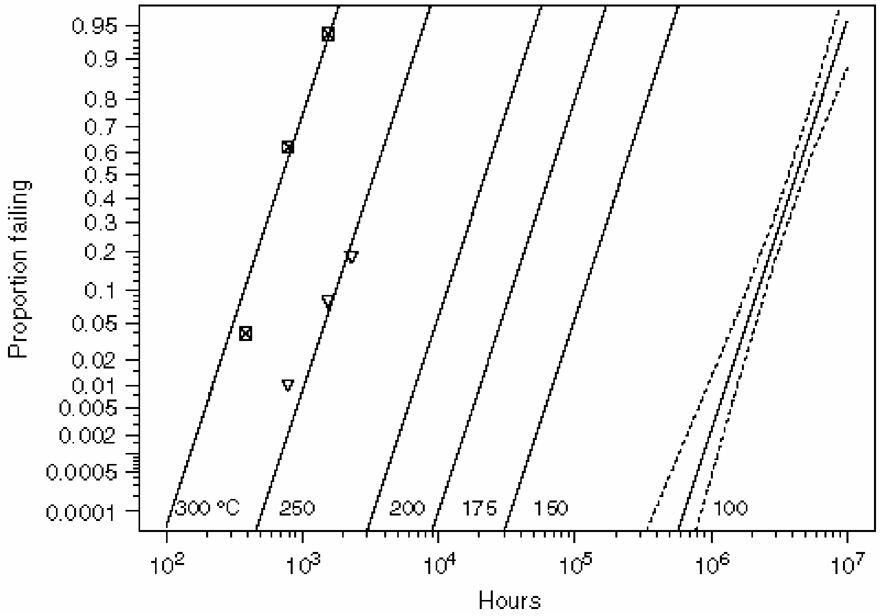
- Introduction
- Some related literature
- General framework
    - Reliability model and experiment
    - Design criterion and its large-sample approximate
    - Design optimization
    - Prior distributions
- Example:
    - Accelerated life test (ALT)

Introduction

- Traditional reliability experiments (non-Bayesian)
  - Parameters considered either fixed/known or totally unknown; no prior in inference

    Example: ALT using temperature





  - Need “planning values” of parameters in design

Introduction

- In most applications, useful but imperfect information is available
  → Bayesian approach
  - In inference, use prior distribution to describe the available info
  - In design
    - Account for the fact that prior will be used in analysis
    - Prior should also be used in planning

    → Preposterior analysis

- Practical issues
  - Reliability experiments are often censored and “accelerated”
  - Typically no close form solution; need numerical optimization
  - Use approximation or simulation

Some Related Literature

- **Non-Bayesian reliability test planning:** Chapter 6 of Nelson (1990), Chapters 10, 20, and 22 of Meeker and Escobar (1998)
- **Bayesian design in reliability test / ALT:** Chaloner and Larntz (1992), Polson (1993), Verdinelli *et al*. (1993), Erkanli and Soyer (2000), Zhang and Meeker (2006)
- **Bayesian design in general and in other problems:** Raiffa and Schlaifer (1961), Chaloner and Larntz (1989), Chaloner and Verdinelli (1995), Clyde *et al*. (1995), Hamada *et al*. (2001)
- **General optimum design methods:** Fedorov (1972), Whittle (1973), Atkinson and Donev (1992)

Reliability Model and Experiment

- A general model
  - Parameter vector $\boldsymbol{\theta}$
  - Interested reliability measure: a scalar $g(\boldsymbol{\theta})$
    - e.g., reliability of a system at a point in time, a quantile of the product life distribution, or the time to first failure, etc
  - Prior distribution $\omega(\boldsymbol{\theta})$
    - Prior in (approximately) independent parameters, then transformed to $\omega(\boldsymbol{\theta})$
    - Shape: typically not highly sensitive
- Experiment
  - Different types: e.g. ALT or degradation tests
  - A particular specified plan: **D**

Design Criterion: Quantifying Precision

- The expected information from an experiment **D**

$$\mathbf{I}_{\boldsymbol{\theta}}(\mathbf{D}) = \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta}, \mathbf{D})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^T} \right)$$

  - Escobar and Meeker (1998) provide algorithms to compute it for different kinds of censored data

- Posterior variance

$$\mathrm{Var}(\boldsymbol{\theta} \mid \mathbf{t}, \mathbf{D}) \approx [\mathbf{S}^{-1} + \hat{\mathbf{I}}_{\boldsymbol{\theta}}(\mathbf{D})]^{-\mathbf{1}}$$

$$\mathrm{Var}(g \mid \mathbf{t}, \mathbf{D}) \approx \left(\overset{\wedge}{\mathbf{g}}' \right)^{\mathrm{T}} \mathrm{Var}(\boldsymbol{\theta} \mid \mathbf{t}, \mathbf{D}) \left(\overset{\wedge}{\mathbf{g}}' \right)$$

- Criterion: preposterior expectation

$$\mathrm{E}_{\mathbf{t}|\mathbf{D}}[\mathrm{Var}(g \mid \mathbf{t}, \mathbf{D})] \approx \int \mathrm{Var}(g \mid \mathbf{t}, \mathbf{D})_{\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}}\, d(\omega(\boldsymbol{\theta}))$$

  - With the dimension of the number of parameters

Design Optimization

- Optimum design
  - Much work in the literature
  - One note: for nonlinear problems like this, the number of design points generally has no theoretical upper bound (e.g. Chapter 19 of Atkinson and Donev 1992 and Section 5.2 of Chaloner and Verdinelli 1995)
    - Sequential numerical search
    - Verify global optimality with General Equivalence Theorem (GET) by Whittle 1973
- More robust design
  - Compromise test plan (e.g. Meeker and Escobar 1998 in non-Bayesian applications; Zhang and Meeker 2006 in Bayesian ALT)

Prior Distributions

- In Bayesian design preposterior analysis, the prior is used twice: posterior calculation and preposterior expectation
  - Typically use the same prior in both places
  - Problem with diffuse/flat prior distributions
    - Technical difficulties with a dataset containing little information
    - Technical difficulties in preposterior expectation
  - An alternative approach: use two different prior distributions
    - Effectively done in some approaches in literature
    - Non-Bayesian design as an extreme

Example

Bayesian Planning in Accelerated life tests

(Zhang and Meeker 2006)

Bayesian ALT Planning Problem

- The model
  - Lifetime distribution: log-location-scale $\Phi(\mu, \sigma)$
  - Acceleration model: $\mu = \gamma_0 + \gamma_1 x, \;\; x \in [x_U, x_H]$
    $\rightarrow$ rescaled $\;\mu = \beta_0 + \beta_1 \xi, \;\; \xi \in [0, 1]$
  - $\boldsymbol{\theta} = (\beta_0, \sigma, \beta_1)^T$
  - $g(\boldsymbol{\theta}) = \log(t_p(x_U)) = \mu_U + \Phi^{-1}(p)\,\sigma = \mathbf{c}^T \boldsymbol{\theta}$
  - $\omega(\boldsymbol{\theta})$ based on independent $\boldsymbol{\theta}^{\Diamond} = (t_{p\Diamond}(x_U), \sigma, -\beta_1)^T$
- Experiment planning
  - **D:** $(\xi_i, \pi_i), \; i = 1, \ldots, k,$ with $\Sigma\, \pi_i = 1$ and $0 \leq \xi_i, \pi_i \leq 1$
  - Typically $\xi_H = 1$

Numerical Example

- Temperature ALT for an adhesive bond
  - Weibull lifetime distribution
  - Arrhenius model: in the range 50°C - 120°C
  - n = 300; type I censoring 183 days
  - Quantity of interest $\log(t_p(x_U))$, $p = 0.10$
  - Prior specification on $\boldsymbol{\theta}^{\Diamond} = (t_{p^\Diamond}(x_U), \sigma, -\beta_1)^T$
    - Independent lognormal distributions
    - $t_{p^\Diamond}(x_U)$: Median = 183, SD = 1630; $p^\Diamond = 0.001$ → diffuse
    - σ: Median = 0.57, SD = 0.20 → moderate
    - $-\beta_1$: Median = 4.62, SD = 0.50 → informative

Two-Point Optimization

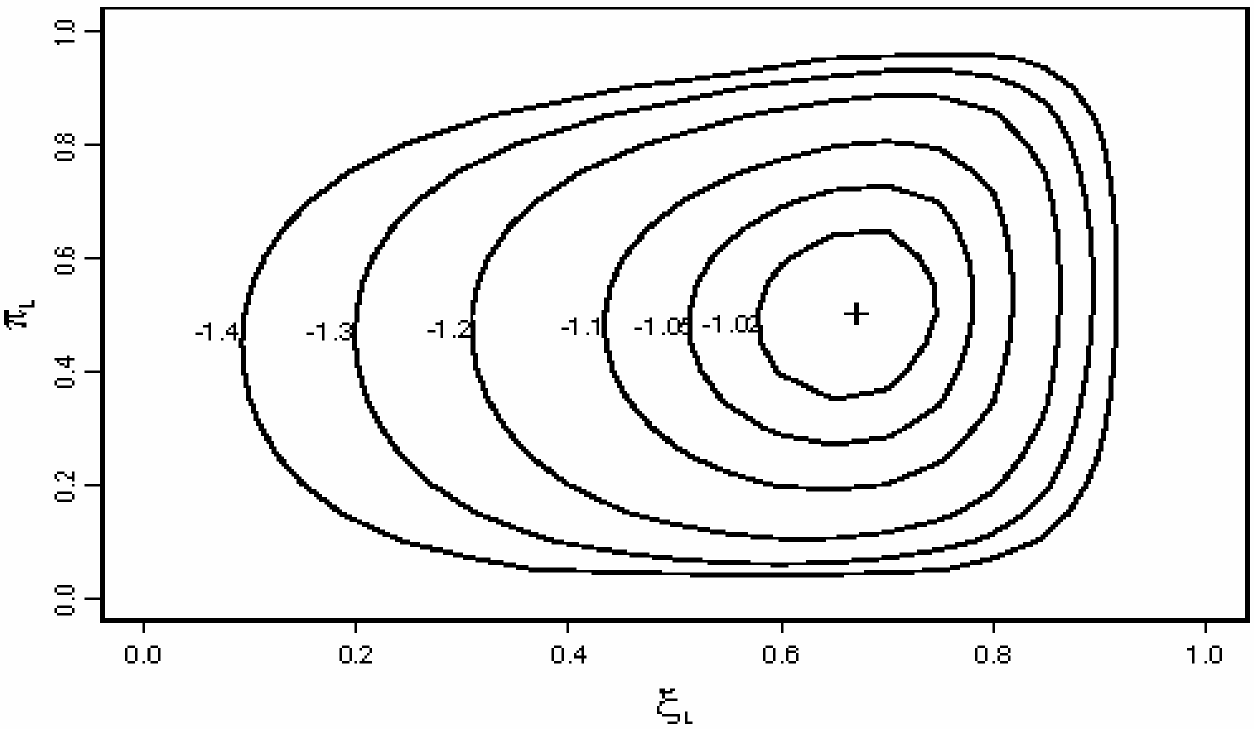


Figure 1: Contour plot of two-point ALT test plans showing the scaled criterion $C(\xi_L, \pi_L)/|C_{\max}|$ as a function of ξ_L and π_L.

Three-Point Optimization

(Collapses to a two-point plan)

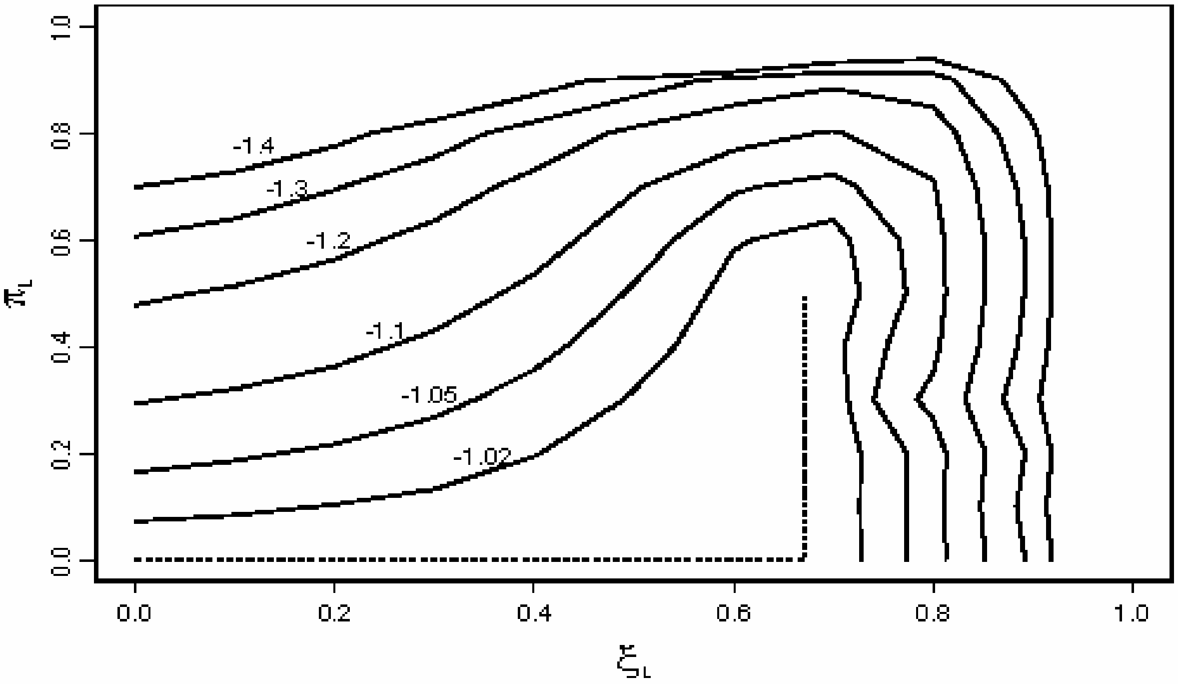


Figure 2: Profile maximum contour plot of three-point ALT test plans showing $C_p(\xi_L, \pi_L)/|C_{\max}|$ as a function of ξ_L and π_L.

Derivative Function (GET)

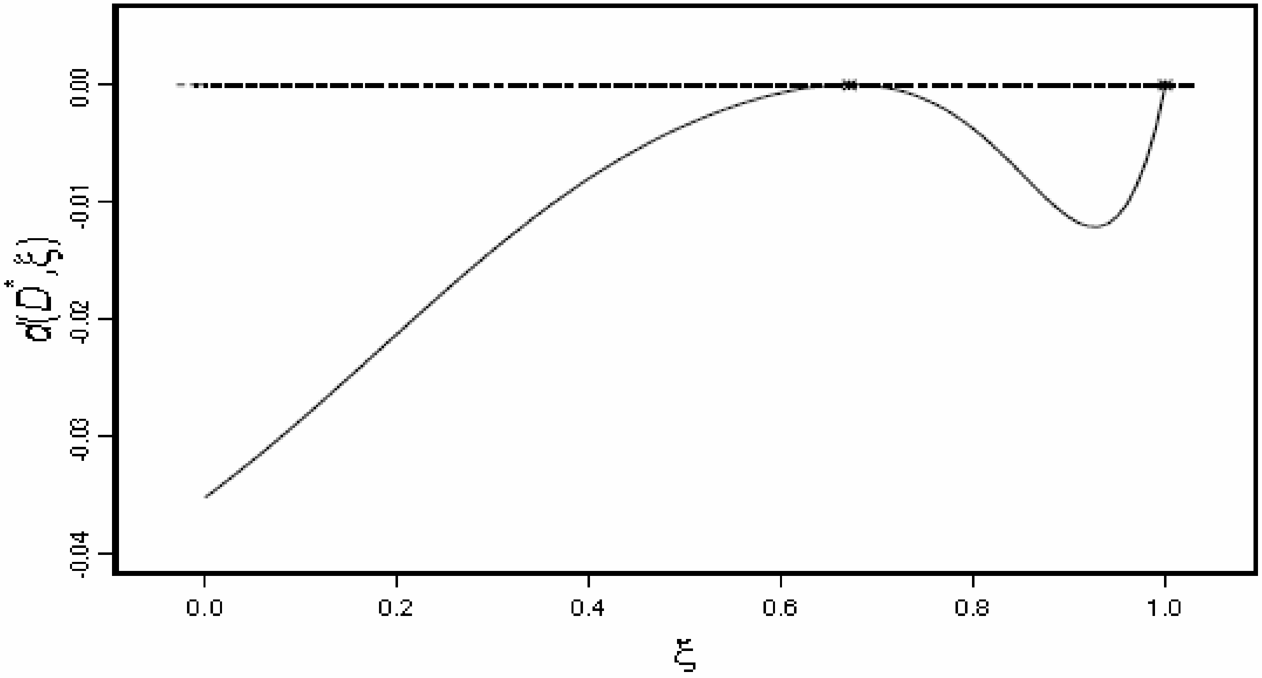
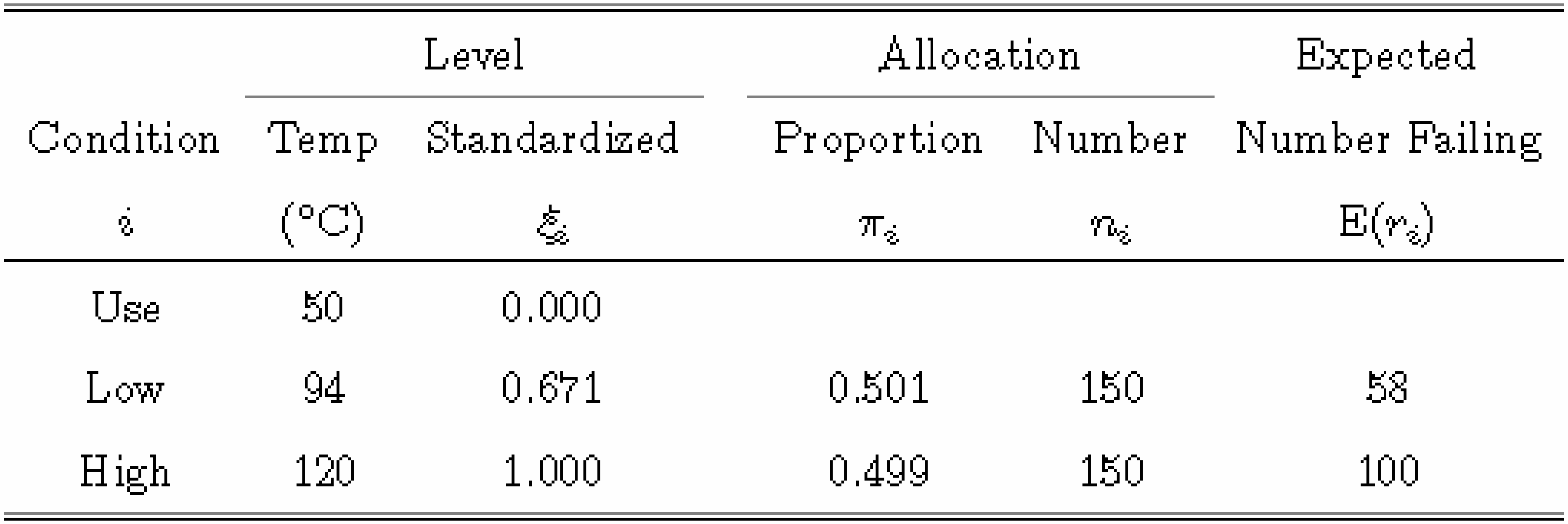


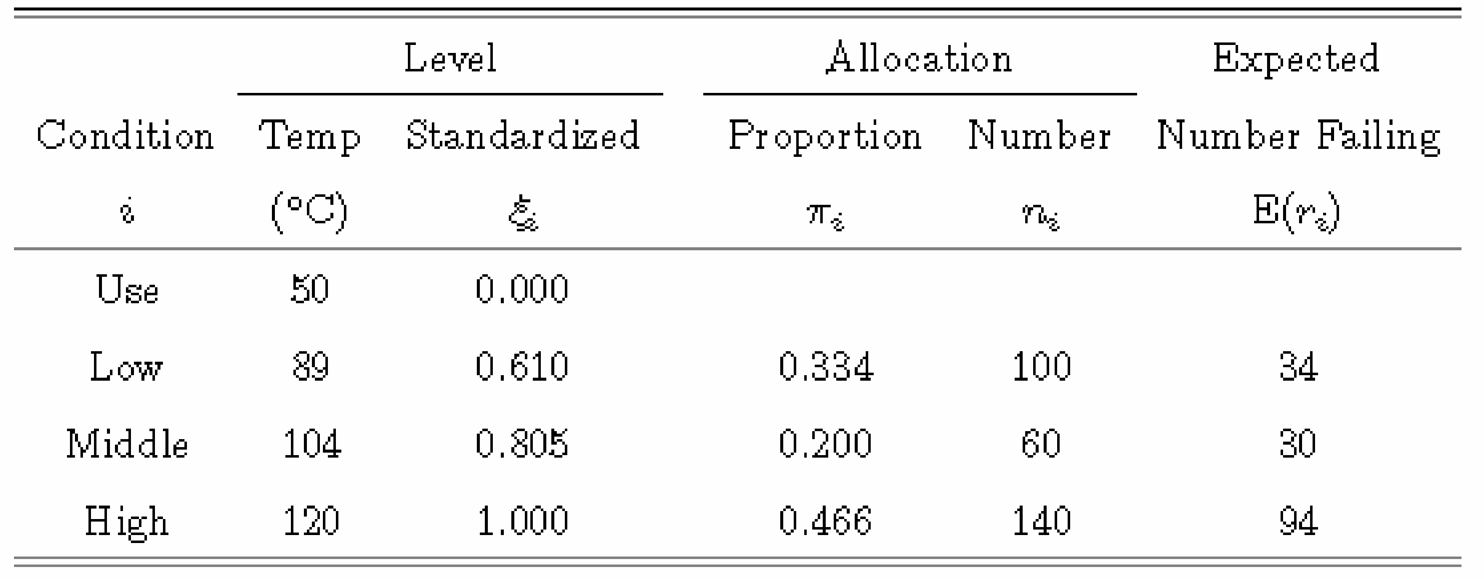
Figure 3: Derivative function $d(D^*, \xi)$ of the criterion (8) at the obtained optimum plan D^*.

Optimum Test Plan



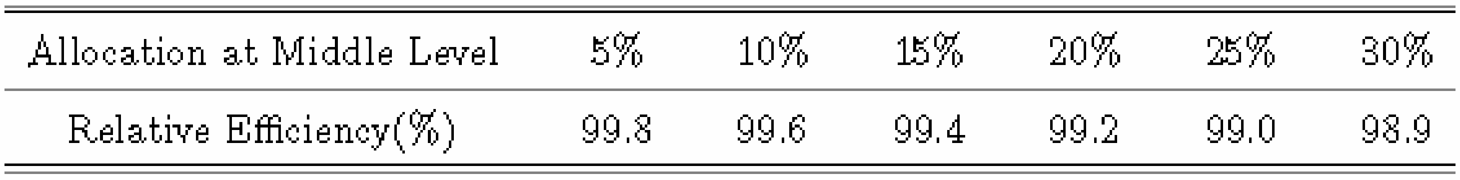
	Level		Allocation		Expected
Condition	Temp	Standardized	Proportion	Number	Number Failing
i	(°C)	ξ_i	π_i	n_i	$\mathrm{E}(r_i)$
Use	50	0.000			
Low	94	0.671	0.501	150	58
High	120	1.000	0.499	150	100

Optimized Compromise Test Plan



	Level		Allocation		Expected
Condition i	Temp (°C)	Standardized ξ_i	Proportion π_i	Number n_i	Number Failing $E(r_i)$
Use	50	0.000			
Low	89	0.610	0.334	100	34
Middle	104	0.805	0.200	60	30
High	120	1.000	0.466	140	94

Relative Efficiency



Allocation at Middle Level	5%	10%	15%	20%	25%	30%
Relative Efficiency(%)	99.8	99.6	99.4	99.2	99.0	98.9

Further Insights

- Effects of changing inputs
  - Amount of prior information
  - Censoring
  - Sample size
- Comparison with Non-Bayesian optimum planning
- Simulation evaluation of a Bayesian plan
  - Simulation based on fixed model parameters
  - Simulation based on simulated model parameters

Summary Remarks

- It makes sense to use Bayesian approach to deal with the often available prior information on model parameters
- A large-sample normal approximation is provided to give an easy to interpret yet useful simplification of the problem