Successive events modeling

Gejza Dohnal

Center for Quality and Reliability of Production
Czech Technical University in Prague

Quality & Productivity Research Conference
IBM T. J. Watson Research Ctr., Yorktown Heights, NY
June 3–5, 2009

Introduction

Failure × **Disaster**

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Failure	×	**Disaster**
random occurrence in time		random occurrence in time
independent recurrence		growing spread
renewal process		domino effect
quick repair or renewal		repair slower than spreading

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accidents, health damage		disease, epidemic
information system failure		system breakdown, piracy attack
motor vehicle accident		traffic collapse
natural accident		natural disaster
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Model → **Prediction** → **Prevention**

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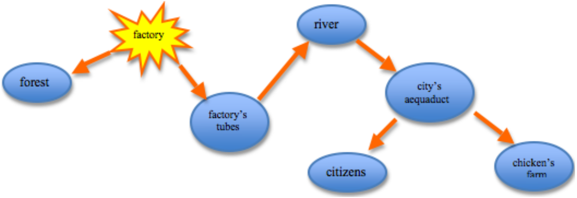
Model → Prediction → Prevention

Preventive Maintenance Policy　　Disaster Recovery Plan

Introduction

The disastrous model (ESREL, September 2008):



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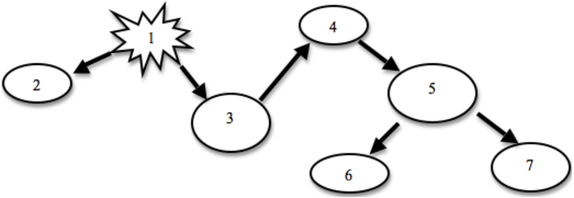


State of the system: $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t)), \quad t \geq 0,$

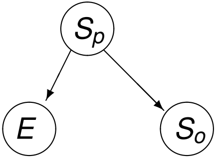
(i) at the beginning (at the time 0), the system is in the state $\omega^0 = (0, 0, \ldots, 0)$,

(ii) the process starts by deterioration of an object i with probability $\pi_i, i = 1, 2, \ldots, n$,

(iii) when at the time t an object i was affected, there was a random time period τ after which the event moved onto some of the unaffected objects,

(iv) states of the system in time t create a stochastic process $\{X(t), t \geq 0\}$ in continuous time. Values of this process lie within the set $\Omega = \{0, 1\}^n$

SET model

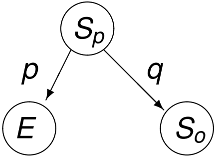


Successive event chain (SEC)

SET model

At any tme $t > 0$, SEC will make decisions on the following three possibilities:

- to leave the state S_p and continue to a offspring event S_o with intensity q
- to finish the sequence with intensity p.
- to stay in the state S_p for a random dwell time T, which has an exponential distribution $\text{Exp}(-(q+p))$



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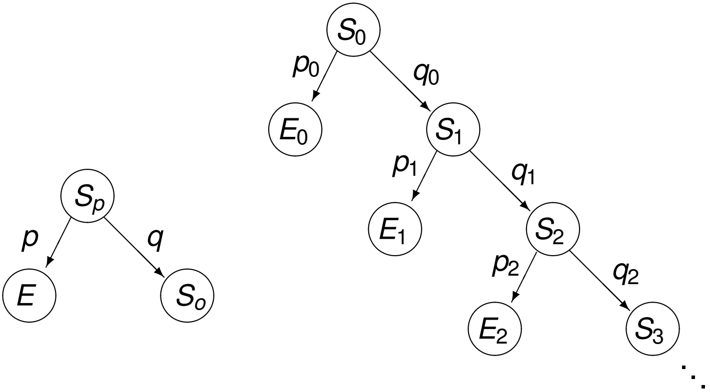


Successive event chain (SEC) – successive event tree (SET)

SET model

$\mathcal{S} = \{S_0, S_1, \ldots, S_n\}$ as the set of successive events,
$\mathcal{E} = \{E_0, E_2, \ldots, E_n\}$ as the set of termination events.

We shall describe the SET as a homogenous Markov process $\{X(t), t \geq 0\}$ with continuous time and a finite set of states $\mathcal{S} \cup \mathcal{E}$.

The process starts in the state S_0 with probability 1.
All states in $\mathcal{S}$ are transient and the states in $\mathcal{E}$ are trapping.

$q_{i,i+1} \quad = q_i$ transition intensity from state S_i to S_{i+1},
$q_{i,n+i+2} = p_i$ transition intensity from S_i to absorbing state E_i.

SET model

At any tme $t > 0$, SEC will make decisions on the following three possibilities:

- to stay in the state S_i for a random dwell time T_i, which has an exponential distribution $\mathrm{Exp}(-(q_i + p_i))$
- to leave the state S_i and continue to a offspring event S_{i+1} with intensity q_i
- to finish the sequence with intensity p_i in a state E_i.

SET model

The transition intensity matrix has the block form

$$Q = \begin{pmatrix} Q_S & Q_E \\ O & O \end{pmatrix} \tag{1}$$

where Q_E is a $(n+1) \times (n+1)$ diagonal matrix whose elements p_i are the intensities of stopping after the ith successive event $S_i,\ i = 0, \dots, n$ and O denotes $(n+1) \times (n+1)$ null matrix. Q_S is the $(n+1) \times (n+1)$ transition intensity matrix from states in $\mathcal{S}$ to states in $\mathcal{S}$ in the form

$$Q_S = \begin{pmatrix} -(q_0 + p_0) & q_0 & 0 & \cdots & 0 \\ 0 & -(q_1 + p_1) & q_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -p_n \end{pmatrix}$$

SET model

Let us consider the embedded Markov chain $\{Y_n\},\ n = 0, 1, \dots, n$. The transition probability matrix of Y in the block form is

$$P = \begin{pmatrix} P_S & P_E \\ O & I \end{pmatrix}$$

where

$$P_S = \begin{pmatrix} 0 & \frac{q_0}{(q_0+p_0)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{q_1}{(q_1+p_1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where P_E is a $(n+1) \times (n+1)$ diagonal matrix whose elements $p_i/(q_i + p_i)$ are the probabilities of stopping after the i-th successive event $S_i,\ i = 0, \dots, n$. O denotes a $(n+1) \times (n+1)$ null matrix and I is the $(n+1) \times (n+1)$ identity matrix.

Proposition

Let $\{X(t), t \geq 0\}$ is a SET with transition intensity matrix Q. Then $X(t)$ stops in the state k with probabilities

$$\pi_0 = \frac{p_0}{p_0+q_0} = 1 - \frac{q_0}{p_0+q_0},$$

$$\pi_k = p_k \cdot \frac{\prod_{j=0}^{k-1} q_j}{\prod_{j=0}^{k}(p_j+q_j)}, \quad k = 1, \ldots, n-1,$$

$$\pi_n = \frac{\prod_{j=0}^{n-1} q_j}{\prod_{j=0}^{n-1}(p_j+q_j)}.$$

The mean time to stopping after the ith state is

$$T_k = \sum_{j=0}^{k} \frac{1}{p_j+q_j}.$$

Example

Let us consider a SET which is assembled from n indentical SECs. This means that we have $q_i = q,\ p_i = p$ for all $i = 0, \dots, n$. Then the probability distribution of terminal states $\mathcal{E}$ has the form

$$\pi_k = \left(\frac{q}{p+q}\right)^{k-1} \left(1 - \frac{q}{p+q}\right),\ 1 \leq k \leq n-1,$$
$$\pi_n = \left(\frac{q}{p+q}\right)^{n-1}$$

and the mean stop-time is $T_k = \frac{k}{p+q}$. The mean "duration" of such finite SET is $\frac{1}{p}\left(1 - \frac{q^n}{(p+q)^n}\right) = \frac{1}{p}(1 - \pi_n)$.

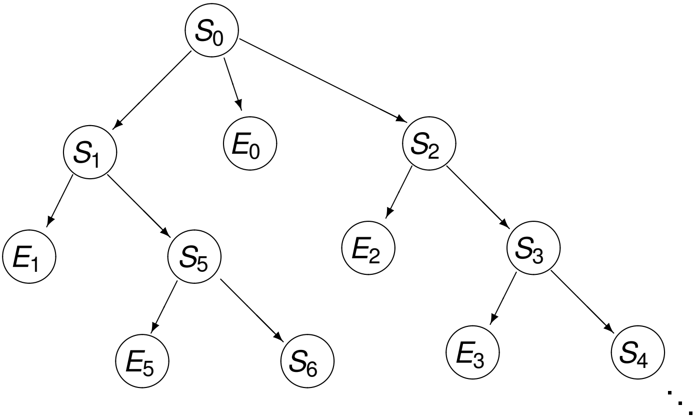
A note: Semi-markovian model

Consider a model in which dwelt times are not necessary exponential. It means that we suppose a set of given distributive functions $F_i(t),\ t \geq 0$ of dwelt times in states $S_i,\ i = 1, \ldots, n$, and the process $X(t)$ to be semi–markovian. When the embedded Markov chain remains the same, the previous Proposition holds but the mean time to stopping will be

$$T_k = \sum_{j=0}^{k} E(\tau_j),$$

where $E(\tau_j) = \int_0^\infty t dF_j(t)$.

One-way branching model



Proposition

Let $\{X(t), t \geq 0\}$ be a SET with $(n \times n)$ transition intensity matrix Q in the previous form where Q_S and Q_E are upper triangular matrices. Denote P the transition probability matrix of an embedded Markov chain.

Then $X(t)$ stops in the state k with probabilities

$$\vec{\pi} = \vec{a} P_S^* P_E,$$

where $\vec{a} = (1, 0, \ldots, 0)$ is a initial probability vector of $X(t)$ and $P_S^* = \sum_{j=1}^{n} P_S^j$.

The mean time to stopping in state E_k is given by the formula

$$T_k = -\vec{a}(Q_S^{(k)})^{-1} \vec{e}\,',$$

where $\vec{e} = (1, 1, \ldots, 1)$ and $Q_S^{(k)}$ equals to the matrix Q_S but diagonal elements are

$$q_{i,i}^{(k)} = q_{i,i} + \sum_{j \neq k} q_{i,j}.$$

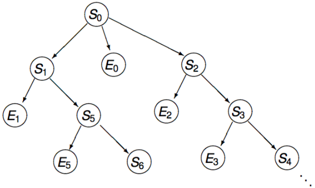
Semi-markovian model

In a semi–markov case, the situation is more complicated.

Suppose that we are able to approximate distribution functions F_j by phase–type distributions with representations $(B_j, b_j^0),\ j = 1, \ldots, n$ (see Neuts, 1975). Then the transition intensity matrice has the form

Conclusions

- There has been proposed the model for successive events spreading in two variants:



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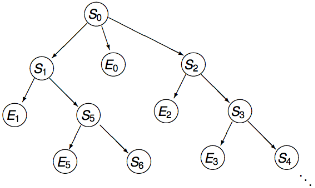

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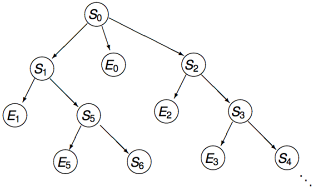

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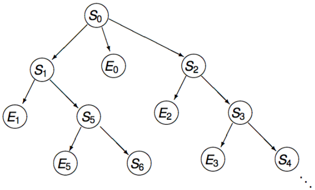

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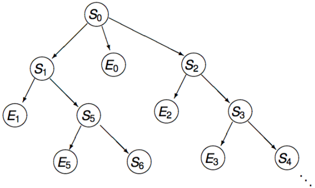

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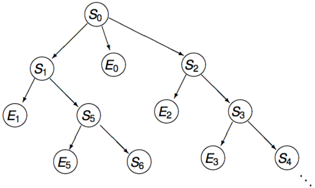

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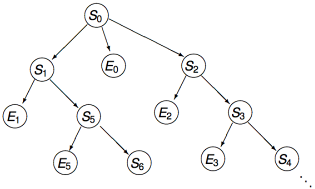

Thanks for your attention.

Gejza Dohnal
Centre for Quality and Reliability of Production, CTU Prague
dohnal@nipax.cz