Estimation and Confidence Intervals for Relative Offset Under Correlated Exponential Network Delays

Jeff Pettyjohn
University of California
Riverside, 92521

* A Collaborative work with Dr. Daniel R. Jeske and Dr. Jun Li
Outline

• Motivation
  – Wireless Geolocation Algorithms
• Notation
  – Offset
  – Timing Message Exchange
• Work from the Literature
• GLMM
  – Maximum Likelihood Estimator
  – Confidence Interval
  – Simulation Results
• Future Work
• References
Motivation: Wireless Geolocation Algorithms

Time of Arrival (TOA) Method

1. Mobile sends Find.Me request at time $T_0$
2. Base station $i$ receives request at time $T_1$ and routes it to the location server
3. Location server
   a) computes $d = (T_1 - T_0) \times c$, for each $i$
   b) computes location prediction for $i \geq 3$
4. Offset between the clocks at the BTSs and the mobile can be a significant source of error in the location prediction.
Offset Notation

\[ C_B(t) = C_A(t) + \theta(t) \]

General Case

\[ C_B(t) = C_A(t) + \theta \]

Constant Offset

Time Showing on Clock

Standard Time, t

Offset can be assumed constant if the adjustment interval is short.
Timing Message Exchanges

To synchronize two clocks $t_A$ and $t_B$, $n$ timing messages are exchanged.
Timing Message Exchanges

A timestamps the $i^{th}$ message $T_i^0$ as it sends it to $B$
When $B$ receives the message, it timestamps it $T_i^1$. 
Due to the offset between $A$ and $B$, $T_i^1 = T_i^0 + d + \theta + e_i^{AB}$
Timing Message Exchanges

\[ T_i^1 = T_i^0 + d + \theta + \epsilon_{iT} \]

\[ T_i^2 \]

NETWORK

\[ B \text{ again timestamps the message } T_i^2 \text{ as it sends it to } A \]
Timing Message Exchanges

When $A$ receives the message, it timestamps it $T_i^3$. 

\[ T_i^1 = T_i^0 + d + \theta + \epsilon_i^{AB} \] 

\[ T_i^3 \]
Timing Message Exchanges

Due to the offset between A and B, $T_i^3 = T_i^2 + d - \theta + \epsilon_{BA}^i$
Each of the $i = 1, \ldots, n$ message exchanges produces four timestamps $(T_i^0, T_i^1, T_i^2, T_i^3)$.
Timing Message Exchanges

Define the one-way transit times as $X_i = T_i^1 - T_i^0$ and $Y_i = T_i^3 - T_i^2$.
Work From the Literature

Paxson (1998) proposes that offset be estimated as

$$\hat{\theta} = \left( \min_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} Y_i \right) / 2 .$$

This estimator is motivated by noting that

$$\min_{1 \leq i \leq n} X_i = d + \theta + \min_{1 \leq i \leq n} e_i^{AB} \approx d + \theta$$

and

$$\min_{1 \leq i \leq n} Y_i = d - \theta + \min_{1 \leq i \leq n} e_i^{BA} \approx d - \theta .$$

Jeske (2005) also showed that when

$$e_i^{AB} \overset{iid}{\sim} \text{Exponential}(\lambda_1) \perp e_i^{BA} \overset{iid}{\sim} \text{Exponential}(\lambda_2)$$

$\hat{\theta}$ is the maximum likelihood estimator (MLE).
Work From the Literature

Under the same exponential delay assumptions, Jeske and Sampath (2003) and Jeske (2006) derived several improvements on Paxson’s estimator:

1. A bootstrap bias correction, $\hat{\theta}^{BC}$
2. A jack-knife bias correction, $\hat{\theta}^{JK}$
3. An o-BLUE, $\tilde{\theta}$
4. A bootstrap bias corrected o-BLUE, $\tilde{\theta}^{BC}$

In simulation studies, under exponential, gamma, and Weibull delay distributions, it was found that in terms of mean squared error (MSE)

$$\tilde{\theta}^{BC} < \tilde{\theta} < \hat{\theta}^{JK} < \hat{\theta}^{BC} < \hat{\theta}.$$  

However, under heavier-tailed distributions such as Pareto, Jeske and Chakravartty (2006) found that $\hat{\theta}^{BC}$ is the most reliable with respect to MSE.
Under the assumption that the delays are from scale families (i.e. Rayleigh distributions), Li, Jeske, and Pettyjohn (2008) developed an approximate confidence interval for $\theta$ based on the likelihood function of the timestamps $\{(T_i^0, T_i^1, T_i^2, T_i^3)\}_{i=1}^n$.

Several non-parametric generalized confidence intervals were also proposed.
Correlated Delays

All of the work in the literature shown assumes that $e_i^{AB}$ and $e_i^{BA}$ are independent.
Correlated Delays

\[ T_i^1 = T_i^0 + d + \theta + e_i^{AB} \]

\[ T_i^2 = T_i^3 + d - \theta + e_i^{BA} \]

However, this is not a realistic assumption
If some factor is influencing uplink delays, it will likely do the same to downlinks
GLMM

It seems more reasonable to allow \( e_i^{AB} \) and \( e_i^{BA} \) to exhibit correlation, but perhaps retraining independence across \((e_i^{AB}, e_i^{BA})\) pairs.

For \( i = 1, \ldots, n \), take \( \alpha_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \), as random delay effects that associate with uplink and downlink delays corresponding to the i-th message exchange.

Assume that conditional on \( \alpha_i \)

\[
e_i^{AB} \overset{ind}{\sim} \text{Exponential}(\lambda_i e^\alpha_{i1}) \perp e_i^{BA} \sim \text{Exponential}(\lambda_2 e^\alpha_{i2}).
\]

This, in effect, is a GLMM.

Note that \( \sigma^2 = 0 \) implies independent uplink and downlink delays.
Under this model, it can be shown that

\[
E \left( \begin{array}{c} X_i \\ Y_i \end{array} \right) = \left( \begin{array}{c} d + \theta + \lambda_1 e^{\sigma^2/2} \\ d - \theta + \lambda_2 e^{\sigma^2/2} \end{array} \right)
\]

and

\[
V \left( \begin{array}{c} X_i \\ Y_i \end{array} \right) = \left( \begin{array}{cc} \lambda_1^2 \left( 2e^{2\sigma^2} - e^{\sigma^2} \right) & \lambda_1 \lambda_2 \left( e^{2\sigma^2} - e^{\sigma^2} \right) \\ \lambda_1 \lambda_2 \left( e^{2\sigma^2} - e^{\sigma^2} \right) & \lambda_2^2 \left( 2e^{2\sigma^2} - e^{\sigma^2} \right) \end{array} \right)
\]

It follows that

\[
Corr(X_i, Y_i) = \frac{e^{\sigma^2} - 1}{2e^{\sigma^2} - 1} \in [0, .5]
\]
**GLMM: MLEs**

With $n$ message exchange, it can be shown that the likelihood function for the set of parameters $\Theta = (\theta, d, \lambda_1, \lambda_2, \sigma)$ is

$$L(\Theta) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} h_i(\alpha_i) \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{\alpha_i^2}{2\sigma^2} \right\} d\alpha_i$$

where

$$h_i(\alpha_i) = \frac{1}{\lambda_1 \lambda_2 e^{2\alpha_i}} \exp \left\{ -\frac{1}{\lambda_1} \left( x_i - d - \theta \right) \right\} \exp \left\{ -\frac{1}{\mu_2} c^{\alpha_i} (y_i - d + \theta) \right\}$$

Fixing $\lambda_1, \lambda_2$ and $\sigma$, it can be shown that the MLEs for $\theta$ and $d$ are

$$\hat{\theta} = \left( \min_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} Y_i \right) / 2 \text{ and } \hat{d} = \left( \min_{1 \leq i \leq n} X_i + \min_{1 \leq i \leq n} Y_i \right) / 2.$$  

Thus Paxon’s estimator is still the MLE under this new model.
GLMM: MLEs

For reasons to become clear later, it is also desired to find MLEs for the set of remaining parameters $\Theta_1 = (\lambda_1, \lambda_2, \sigma)$. Substituting the MLEs $\hat{\theta}$ and $\hat{d}$ into the original likelihood function yields a reduced likelihood function

$$L(\Theta_1) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} h^*_i(\alpha_i) \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{\alpha_i^2}{2\sigma^2} \right\} d\alpha_i$$

where

$$h^*_i(\alpha_i) = \frac{1}{\lambda_1 \lambda_2 e^{2\alpha_i}} \exp \left\{ -\frac{1}{\lambda_1} \left( x_i - \hat{d} - \hat{\theta} \right) \right\} \exp \left\{ -\frac{1}{\mu_2} \left( y_i - \hat{d} + \hat{\theta} \right) \right\}$$

The integration to numerically obtain the MLEs is rather difficult. However, using Gauss-Hermite quadrature, an approximated likelihood

$$L(\Theta_1) \approx \prod_{i=1}^{n} \left( \sum_{k=1}^{K} \frac{w_k}{\sqrt{\pi}} h^*_i \left( \sqrt{2} v_k \right) \right)$$

can be used, where $w_k$ and $v_k$ are the quadrature weights and points.
GLMM: CIs

It is the goal to derive a confidence interval for \( \theta \). Examining the MLE \( \hat{\theta} \) it can be seen that

\[
\hat{\theta} = \left( \min_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} Y_i \right) / 2 \\
= \left( d + \theta + \min_{1 \leq i \leq n} e_i^{AB} - d + \theta - \min_{1 \leq i \leq n} e_i^{BA} \right) / 2 \\
= \theta + \left( \min_{1 \leq i \leq n} e_i^{AB} - \min_{1 \leq i \leq n} e_i^{BA} \right) / 2
\]

From this, it can be shown that the cdf of \( \hat{\theta} \) is

\[
F_{\hat{\theta}}(z) = \begin{cases} \\
\frac{\lambda_1}{\lambda_1 + \lambda_2} \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} e^{2\lambda_2 (z-\theta e^{\alpha_i})} \frac{1}{\sqrt{2\pi \sigma}} e^{-\alpha_i^2/2\sigma^2} d\alpha_i \right\} & z \leq \theta \\
1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} e^{-2\lambda_1 (z-\theta e^{\alpha_i})} \frac{1}{\sqrt{2\pi \sigma}} e^{-\alpha_i^2/2\sigma^2} d\alpha_i \right\} & z > \theta
\end{cases}
\]
GLMM: CIs

As before with the likelihood function, the integration involved in computing the cdf is quite difficult. However, again using Gauss-Hermite quadrature the cdf can be closely approximated as

\[
F_{\theta}(z) \approx \begin{cases} 
\frac{\lambda_1}{\lambda_1 + \lambda_2} \left\{ \sum_{k=1}^{K} \frac{w_k}{\sqrt{\pi}} e^{2\lambda_2 (z-\theta) e^{\lambda_2 \sigma_k}} \right\}^n & z \leq \theta \\
1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \left\{ \sum_{k=1}^{K} \frac{w_k}{\sqrt{\pi}} e^{-2\lambda_1 (z-\theta) e^{\lambda_1 \sigma_k}} \right\}^n & z > \theta 
\end{cases}
\]

where for \( k = 1, ..., K \), \( w_k \) and \( \nu_k \) are respectively the quadrature weights and evalution points.
GLMM: CIs

To use this information to form a confidence interval for $\theta$, define $U = \hat{\theta} - \theta$ and its cdf as $G(u \mid \lambda_1, \lambda_2, \sigma)$. If $\lambda_1$, $\lambda_2$, and $\sigma$ are known, then

$$G(u \mid \lambda_1, \lambda_2, \sigma) \sim Uniform(0,1).$$

Thus if a nominal coverage level $(1 - \alpha)$ is specified such that

$$P(G_U(u \mid \lambda_1, \lambda_2, \sigma) \in (\alpha / 2, 1 - \alpha / 2)) = 1 - \alpha$$

it can be inverted for $u$ to get a $(1 - \alpha)100\%$ confidence interval

$$\left(\hat{\theta} - G^{-1}(1 - \alpha / 2 \mid \lambda_1, \lambda_2, \sigma), \hat{\theta} - G^{-1}(\alpha / 2 \mid \lambda_1, \lambda_2, \sigma)\right).$$

However, $\lambda_1$, $\lambda_2$, and $\sigma$ are unknown, but it conjectured that

$$\left(\hat{\theta} - G^{-1}(1 - \alpha / 2 \mid \hat{\lambda}_1, \hat{\lambda}_2, \hat{\sigma}), \hat{\theta} - G^{-1}(\alpha / 2 \mid \hat{\lambda}_1, \hat{\lambda}_2, \hat{\sigma})\right)$$

is an approximate $(1 - \alpha)100\%$ confidence interval for $\theta$. 
Initial Results

To assess the quality of the approximate confidence interval, simulations were performed in which 10,000 datasets were generated under the assumptions that $\theta = 0$, $d = 0$, $\lambda_1 = 1$, and $\lambda_2 = 5$. Values of $\sigma$ were selected in order to vary correlation from 0 to .5.

The nominal coverage level $(1 - \alpha)$ was selected to be .90 and the performance of the true (cdf) confidence interval was evaluated along with the approximate (acdf) confidence interval.

Tables 1 and 2 show the results to evaluate the frequentist coverage probability and the expected interval width.
# Initial Results: Coverage

**Table 1.** Coverage of Nominal 90\% Confidence Intervals

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Interval</th>
<th>Sample Size</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>20</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>CDF</td>
<td>.900</td>
<td>.900</td>
<td>.900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.872</td>
<td>.882</td>
<td>.891</td>
<td></td>
</tr>
<tr>
<td>.25</td>
<td>CDF</td>
<td>.900</td>
<td>.900</td>
<td>.900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.870</td>
<td>.881</td>
<td>.898</td>
<td></td>
</tr>
<tr>
<td>.50</td>
<td>CDF</td>
<td>.900</td>
<td>.900</td>
<td>.900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.881</td>
<td>.888</td>
<td>.899</td>
<td></td>
</tr>
</tbody>
</table>
Initial Results: Expected Width

Table 2. Expected Width of Nominal 90% Confidence Intervals

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Interval</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>CDF</td>
<td>.230</td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.211</td>
</tr>
<tr>
<td>.25</td>
<td>CDF</td>
<td>.225</td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.211</td>
</tr>
<tr>
<td>.50</td>
<td>CDF</td>
<td>.0838</td>
</tr>
<tr>
<td></td>
<td>ACDF</td>
<td>.0785</td>
</tr>
</tbody>
</table>
Future Work

1. Improve the Paxson estimator under the GLMM
   a) Bootstrap bias correction
   b) Jack-knife bias correction
   c) MVUE

2. Improve the GLMM approximate confidence interval
   a) Bootstrap correction for coverage
   b) Best width interval

3. Change the model to allow for correlation > .5
   a) Assume that $\alpha_i$ are iid $N(\mu, \sigma^2)$.
   b) Assume that $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ are iid bivariate Normal and that the uplink and downlink delays are independent conditional on $(\alpha_{i1}, \alpha_{i2})$
   c) Assume that the uplink and downlink errors are bivariate Exponential
References


