Optimal Design of Accelerated Life Tests with Multiple Stresses

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Outline

- Motivation for ALT with multiple stresses
- Review current work on multiple stresses ALT
- Optimal Latin hypercube designs for ALT
- Modified simulated annealing algorithm
- Numerical examples
- Summary

Motivation for ALT with Multiple Stresses

- Definition: *ALT is conducted to quickly obtain failure times in order to evaluate and predict reliability at normal operating conditions*

- Most of the previous work on designing ALT plans uses single stress
- For highly reliable products, it is difficult to obtain failures in a short time with single stress
- Reliability may depend on several stresses operating simultaneously.
- Multiple stresses represent realistic field conditions

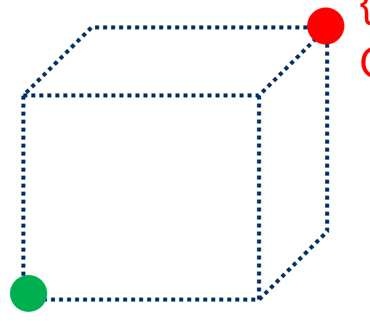
Current Work on Multiple Stresses ALT

- Two stresses and two levels for each stress:
  - Nelson (1990) –- one accelerated stress and one other factor
  - Escobar and Meeker (1995) –- compromise plan
  - Park and Yum (1996) –- factorial design
  - Elsayed and Zhang (2009) –- factorial design
- Difficult to extend existing methods, because:
  - Multiple stresses and levels can result a huge number of stress-level combinations (designs).
  - With limited number of test units and time, how to choose the best or a small number of optimal designs?
  - How to spread the design points across the design region to avoid extreme conditions?

Proposed Solution – Latin Hypercube Design

- Latin hypercube design (n, k):
  - An *LHD* with k factors and n levels (design points).
  - Each factor has n levels taken values in $\{1, 2, \ldots, n\}$
  - Example, *LHD*(5, 3) :

Volt	Temp	Humidity
1	3	4
2	5	1
3	1	5
4	4	2
5	2	3



- Advantages:
  - Three or more factors and levels can be investigated
  - Optimal *LHD* can avoid extreme stress-level combinations
  - Significantly reduce the stress-level combinations: n^k to n.

Optimal *LHD* Criterion

- Inter-site distance: $d(\mathbf{s},\mathbf{t}) = \left\{ \sum_{j=1}^{k} \left| s_j - t_j \right|^q \right\}^{1/q}$

  q=1,2,∞, correspond to rectangular, Euclidean, and infinite distance

- Example:

$LHD(n,k)$:

$LHD(5,3)$:

$$\begin{array}{ccc} \boxed{1\ \ 3\ \ 4} & \longrightarrow & \mathbf{s} \\ \boxed{2\ \ 5\ \ 1} & \longrightarrow & \mathbf{t} \\ 3\ \ 1\ \ 5 & & \\ 4\ \ 4\ \ 2 & & \\ 5\ \ 2\ \ 3 & & \end{array}$$

$\binom{n}{2} = \binom{5}{2} = 10$ pairs of inter-site distance

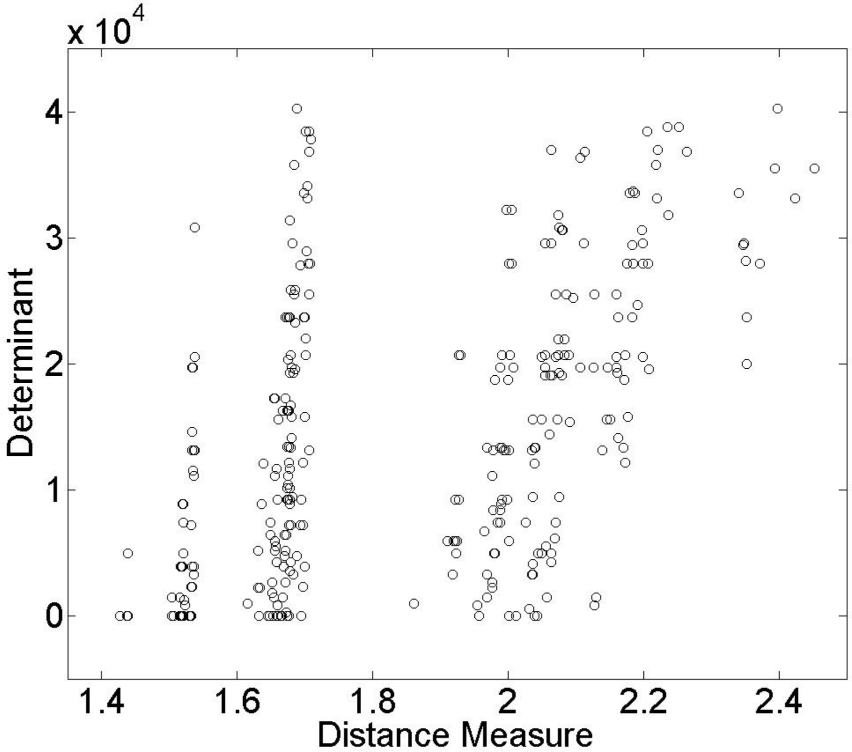
- Morris and Mitchell (1995) proposed maximum minimum inter-site distance criterion:

$$\phi_p = \left(\sum_{i=1}^{\binom{n}{2}} \frac{1}{d_i^p} \right)^{1/p}$$

- Spread the design points across the design region

Optimal ALT Criterion

- *D*-optimality:
  - Ensure the accuracy of parameter estimation
  - Maximize determinant of Fisher information matrix of maximum likelihood estimate
- *D*-optimality may not lead to optimal distance measure



Objective Function

- A multi-objective formulation:

$$\text{Min} \quad -w\frac{\det\left(F_s\left(\mathbf{X}\right)\right)}{Det_U}-\left(1-w\right)\frac{\Phi_p\left(\mathbf{X}\right)-\Phi_{p,L}}{\Phi_{p,U}-\Phi_{p,L}}, \quad w\in\left[0,1\right]$$

  - w is the weight
  - $\mathbf{X}$ is the design matrix
  - $\det\left(F_s\left(\mathbf{X}\right)\right)$ is the determinant of the Fisher information matrix
  - $\Phi_p = 1/\phi_p$ is the distance measure
- Normalize the distance and *D*-optimality criteria:
  - Det_U is the upper bound of the determinant
  - $\Phi_{p,U}$ and $\Phi_{p,L}$ are the upper and lower bound of the distance measure

Propositions

- Focus on three stresses
  - Concurrently applying too many stresses is very demanding for test equipment
  - Extension to more stresses is possible

- For a $LHD(n,3)$, $0 \le \det(F_s) \le Det_U$

$$Det_U = n^4 (n-1)^3 (n+1)^3 \left(n\pi^2 + 2n\gamma(1-\gamma)\right) \big/ 10368,\ n > 1$$

- For a $LHD(n,k)$, $\Phi_{p,L} \le \Phi_p \le \Phi_{p,U}$, $q, p \in [1, \infty)$

$$\Phi_{p,U} = \left[\sqrt{kn(n+1)/6} \Big/ \binom{n}{2}^{1/p} \right] \quad \Phi_{p,L} = \left(\sum_{i=1}^{n-1} \frac{(n-i)}{k^{p/2} \cdot i^p} \right)^{(-1/p)}$$

Proofs are given in our working paper

Log-location Scale Model

- Log times to failure $Y = \ln(T)$ is assumed to follow

$$f_Y = \left(y;\mu(\boldsymbol{\beta}\mathbf{x}),\sigma\right) = \frac{1}{\sigma}\exp\left[\left(\frac{y-\mu(\boldsymbol{\beta}\mathbf{x})}{\sigma}\right) - \exp\left(\frac{y-\mu(\boldsymbol{\beta}\mathbf{x})}{\sigma}\right)\right] \quad -\infty < y < \infty$$

- The location parameter depends on the stresses through

$$\mu(\boldsymbol{\beta}\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$$

- Transformation $Y = \mu(\boldsymbol{\beta}\mathbf{x}) + \sigma Z$
- Extreme value distribution $f_Z(z) = \exp\left(z - \exp(z)\right), \quad -\infty < z < \infty$
- Log likelihood function of failures observed in the m^{th} run

$$l_m = \ln L_m\left(\beta_0,\ldots,\beta_k,\sigma\right) = -N\pi_m \ln(\sigma) + \sum_{j=1}^{(N\pi_m)} z_{m,j} - \sum_{j=1}^{(N\pi_m)} \exp\left(z_{m,j}\right)$$

Fisher Information Matrix

- Assume equal proportion of test units for each design point

$$F = N\pi_1 \sum_j F_{1,j} + N\pi_2 \sum_j F_{2,j} + \ldots + N\pi_n \sum_j F_{n,j}$$

$$F = \frac{N}{n\sigma^2} \begin{bmatrix} n & \sum_{m=1}^{n} x_{1,m} & \cdots & \cdots & \sum_{m=1}^{n} x_{k,m} & n(1-\gamma) \\ \sum_{m=1}^{n} x_{1,m} & \sum_{m=1}^{n} x_{1,m}^2 & \sum_{m=1}^{n} x_{1,m} x_{2,m} & \cdots & \sum_{m=1}^{n} x_{1,m} x_{k,m} & (1-\gamma)\sum_{m=1}^{n} x_{1,m} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \sum_{m=1}^{n} x_{k,m} & \sum_{m=1}^{n} x_{1,m} x_{k,m} & & & \sum_{m=1}^{n} x_{k,m}^2 & (1-\gamma)\sum_{m=1}^{n} x_{k,m} \\ n(1-\gamma) & (1-\gamma)\sum_{m=1}^{n} x_{1,m} & \cdots & \cdots & (1-\gamma)\sum_{m=1}^{n} x_{k,m} & n\left[\frac{\pi^2}{6} + (\gamma-1)^2\right] \end{bmatrix}$$

where $\gamma = 0.5777215$ is Euler's constant.

Generation of *LHD*

- For the distance measure ϕ_p , Morris and Mitchell (1995) used simulated annealing (SA) to generate *LHDs:*
  - Begin with a random chosen *LHD*
  - Perturb the current design *D* by interchanging two randomly chosen elements within a randomly chosen column of the design matrix
  - If the perturbed matrix D_{try} leads to an improvement, it is adopted as the new current design from which the next perturbation is generated
  - Otherwise, a replacement is made with probability

$$\pi = \exp\left\{-\left[\phi\left(D_{try}\right) - \phi\left(D\right)\right]\big/ t\right\}$$

    where *t* is the annealing temperature
  - The algorithm stops either after given iterations or a tolerance criterion is met

Modified SA to Generate *LHD* for ALT

- Choose the elements to interchange based on probabilities
  - $LHD(n,3)$: $$\begin{bmatrix} 1 & x_{2,1} & x_{3,1} \\ 2 & \vdots & \vdots \\ \vdots & & \\ n & x_{2,n} & x_{3,n} \end{bmatrix}$$
  - Maximum determinant of the Fisher information matrix is obtained when $a_1 = \sum_{m=1}^{n} x_{1,m} x_{2,m} = n(n+1)^2/4$ and $a_2 = \sum_{m=1}^{n} x_{1,m} x_{3,m} = n(n+1)^2/4$
  - Choose column based on probability $$\left|a_1 - n(n+1)^2/4\right|^p \Big/ \sum_{i=1,2} \left|a_i - n(n+1)^2/4\right|^p,\ p \in [1,\infty)$$
  - Choose row based on probability $\Phi_{p,i}^p \Big/ \sum_{i=1}^{n} \Phi_{p,i}^p$ (Joseph and Hung, 2008 proposed $\phi_{p,i}^p \Big/ \sum_{i=1}^{n} \phi_{p,i}^p$)

Numerical Example 1 – *LHD* (5,3)

- Testing stress levels for each factor:

LHD	1	2	3	4	5
Volts/cm^2	12	16	20	24	28
Temperature (ºC)	-40	0	40	80	120
Relative Humidity(RH)	20	30	40	50	60

- Set w = 0.5

(p,q)	(5,1)						(5,2)					
	LHD			Volts/cm^2	ºC	RH	*LHD*			Volts/cm^2	ºC	RH
Complete enumeration & Modified SA	1	4	2	12	80	30	1	4	2	12	80	30
	2	3	5	16	40	60	2	1	3	16	-40	40
	3	2	1	20	0	20	3	5	4	20	120	50
	4	1	4	24	-40	50	4	2	5	24	0	60
	5	5	3	28	120	40	5	3	1	28	40	20

Numerical Example 1 – Simulation

- Given parameter values:

$$\boldsymbol{\beta} = \begin{bmatrix} 423 & -12.6 & 6.2 & -3.7 \end{bmatrix} \qquad \sigma = 0.94$$

- Stress settings: optimal design from (p, q) = (5, 2)
- Generate random failure times:

$$F_z = 1 - \exp\left(-\exp(z)\right) \qquad z = \frac{\ln(t_i) - \left(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3\right)}{\sigma}$$

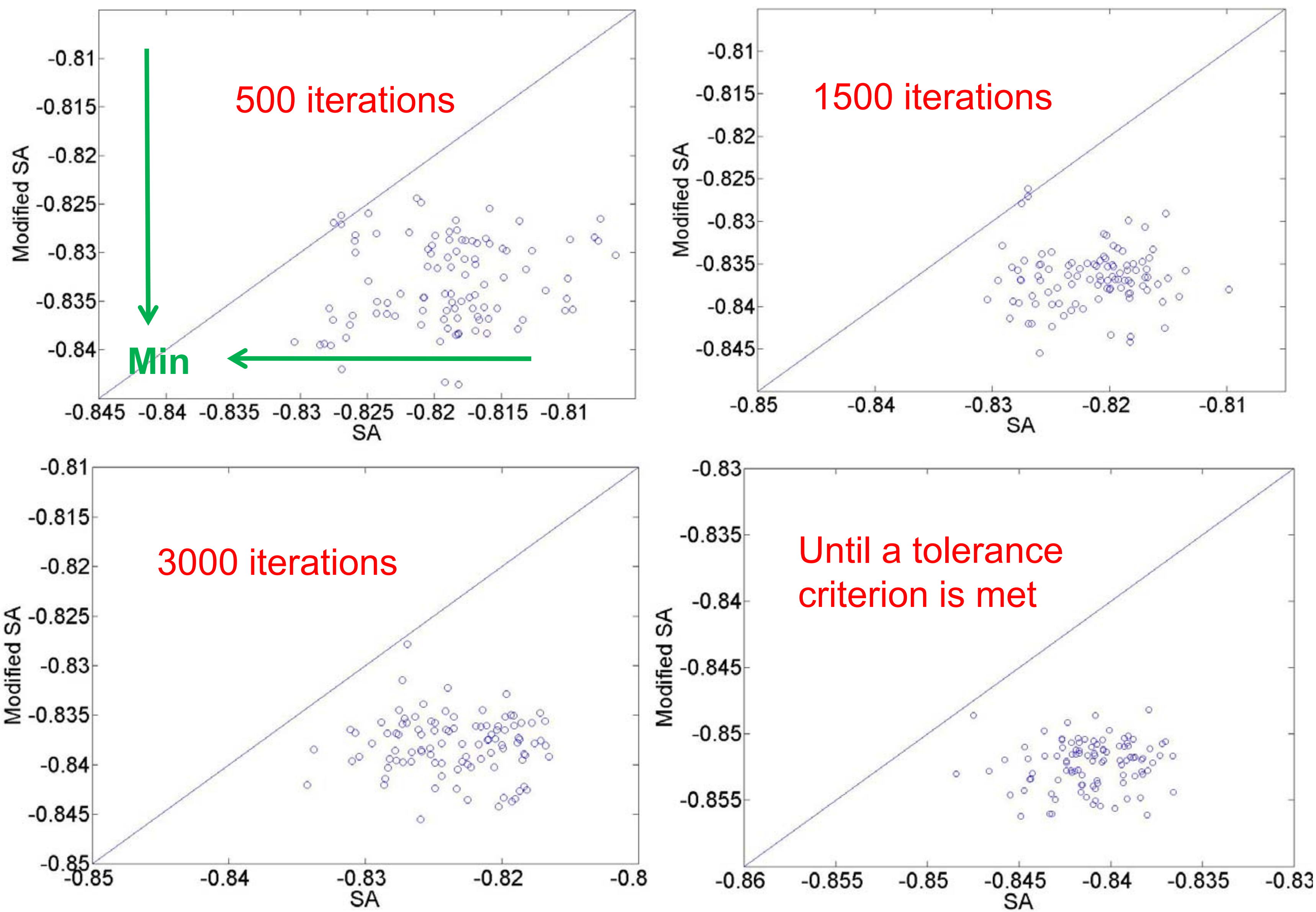
$$t_i = \exp\left[\left(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3\right) + \sigma \ln\left[-\ln\left(1 - rand(i)\right)\right]\right]$$

- Maximum likelihood estimate (repeat 100 times):

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 422.9 & -12.6 & 6.19 & -3.69 \end{bmatrix} \qquad \sigma = 0.91$$

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\beta}} = \begin{bmatrix} 0.66 & -0.02 & 0.2 & -0.006 \end{bmatrix} \qquad \sigma_{\sigma} = 0.08$$

Numerical Example 2 – *LHD* (25, 3)



Numerical Example 2 – 300 Times Simulation

- Count smaller objective function values from MSA than those from SA

Stop Criterion	***LHD* (10, 3)**		***LHD* (25, 3)**	
	***q* = 1**	***q* = 2**	***q* = 1**	***q* = 2**
500 iter.	85.1%	90.2%	98%	95%
1500 iter.	88.3%	91%	98.5%	96.7%
3000 iter.	92%	92.5%	99.33%	98%
Tolerance	100%	100%	100%	100%

- Compare MSA and SA in terms of objective function values

LHD (10,3)	Min		Max		Mean	
$(p, q) = (15, 2)$	MSA	SA	MSA	SA	MSA	SA
500 iter.	-.8843	-.8741	-.8493	-.8273	-.8673	-.8529
1500 iter.	-.8891	-.8794	-.8557	-.8375	-.8731	-.8606
3000 iter.	-.8950	-.8825	-.8654	-.8484	-.8756	-.8657
tolerance	-.8918	-.8833	-.8756	-.8633	-.8821	-.8746

LHD (25,3)	Min		Max		Mean	
$(p, q) = (15, 2)$	MSA	SA	MSA	SA	MSA	SA
500 iter.	-.8500	-.8412	-.8298	-.8100	-.8395	-.8261
1500 iter.	-.8500	-.8423	-.8355	-.8112	-.8424	-.8287
3000 iter.	-.8519	-.8437	-.8360	-.8245	-.8433	-.8318
tolerance	-.8525	-.8473	-.8437	-.8338	-.8486	-.8386

Summary

- Multiple stresses ALT plan is developed using Latin hypercube design, such that the stress-level combinations are dramatically reduced
- The *D*-optimality and maximin distance measures are normalized and combined to generate desired *LHD* for ALT
- Simulated annealing algorithm is modified to efficiently generate optimal *LHDs*.
- Numerical examples validate the feasibility and efficiency of the proposed methodology